ISRAEL JOURNAL OF MATHEMATICS, Vol. 20, No. 2, 1975

SOME MEASURE THEORETIC RESULTS IN EFFECTIVE DESCRIPTIVE SET THEORY

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ABSTRACT

Assuming projective determinacy when it is needed, we prove some structure theorems in the measure theory and the category theory of the analytical hierarchy.

The first studies of measure theory in effective descriptive set theory were done by Spector [12] Sacks [10] and Tanaka [14]; similarly, problems related to category have been studied by Feferman, Hinman [3], and Thomason [15]. Recent work on the consequences of the axiom of projective determinacy have made it possible to carry the theory from the first level of the analytical hierarchy up to an arbitrary level. This has been done by Kechris, whose results have appeared in a nice and comprehensive article on the subject [5]. The present work must be viewed as a supplement to Kechris' article. In the first part, we prove a theorem (due independently to Kechris and the author) asserting the existence of largest Π^1_{2n+1} and Σ^1_{2n} sets of zero measure (resp. of the first category), and, without using determinacy, we study the largest Π^1_1 and Σ^1_2 sets of zero measure.[†] In the second part, we discuss the structure of "large" Π^1_{2n+1} and Σ^1_{2n} sets. Especially, we give some additional information on the approximation of these sets. Another result of interest which we prove is the existence, for any Π^1_{2n+1} (or Σ^1_{2n}) set A, of a largest Π^1_{2n+1} (or Σ^1_{2n}) set A', $A \subseteq A'$ with the same measure.

We wish to thank Profs. D. A. Martin and R. M. Solovay for valuable conversations, which aroused our interest in descriptive set theory, and Dr. A. S. Kechris for a stimulating correspondence.

0. Definitions and notations

In the sequel, we will follow the common notations of modern descriptive set theory. We make a short review of them and recall the basic tools and results

[†] The results in this part have been announced in [13].

Received March 21, 1974

we will use. For more details the reader is referred to [5] and [6]. ω is the set of nonnegative integers and $\mathscr{R} = {}^{\omega}\omega$ is the set of mappings from ω into ω; elements of ${}^{\omega}\omega$ are called "reals". ${}^{\omega}\omega$ is endowed with the usual product topology: a system of basic open sets for this topology is the set of open sets of the form $\hat{s}$ where $\hat{s} = (\alpha : \bar{\alpha}(n) = s)$ and s is a finite sequence of integers of length n. Actually, this topology is induced by a metric and therefore it makes sense to develop the theory of Baire category. If μ_0 is the usual product measure on ${}^{\omega}2$, a measure μ on ${}^{\omega}\omega$ is defined by $\mu(X) = \mu_0(X \cap {}^{\omega}2)$ for any Borel set X.

We are interested into properties of subsets of $\omega^k \times \mathscr{R}^m$ where k and m are integers; we call these subsets "pointsets". A set of pointsets is a "pointclass". We will consider especially the analytical pointclasses Σ^1_n and Π^1_n.

Definition 1 (Moschovakis). Let Γ be a pointclass, A a pointset in Γ; a Γ-norm on A is a mapping ϕ from A onto an ordinal, for which there exist two relations in Γ, R and S, such that if β is an element of A

$$(\alpha \in A \not\subset \phi(\alpha) \leq \phi(\beta)) \leftrightarrow R(\alpha,\beta) \leftrightarrow \neg S(\alpha,\beta);$$

we say R and S define ϕ on A.

The relation on $A \leq_\phi$ such that $\alpha \leq_\phi \beta \leftrightarrow \phi(\alpha) \leq \phi(\beta)$ is a *prewellordering* (i.e. a well-founded, connected, preordering)

Definition 2 (Moschovakis). Let Γ be a pointclass, A a pointset in Γ; a Γ-scale on A is a sequence of Γ norms $(\varphi_n)_{n\in\omega}$ on A, uniformly defined by relations in Γ, $R(n, \alpha, \beta)$, $S(n, \alpha, \beta)$ and such that — if $(\alpha_i)_{i\in\omega}$ is a sequence in A with

$$\lim_{i\to\infty} \alpha_i = \alpha \quad \text{and} \quad \phi_k(\alpha_i) = \lambda_k$$

for large enough i, then

$$\alpha \in A \text{ and } \phi_k(\alpha) \leq \lambda_k.$$

Projective Determinacy is the hypothesis that every projective set is determined. For information about determinacy we refer the reader to the survey article [2]. To read the present paper, it is enough to know the following consequences of the axiom of projective determinacy:

THEOREM 1. *Assume projective determinacy, then*:

(i) (Mycielski, Swierzkowski [9]). *Every projective set is Lebesgue measurable (resp. has the Baire property)*;

(ii) (Martin [7], Moschovakis [1]) *Every* Σ^1_{2n} *(resp* Π^1_{2n+1}*) set has a* Σ^1_{2n} *(resp.* Π^1_{2n+1}*)*- norm;

(iii) (Moschovakis [8]). *Every* Σ^1_{2n} *(resp.* Π^1_{2n+1}*) set has a* Σ^1_{2n} *(resp.* Π^1_{2n+1}*)- scale*;

(iv) (Kechris [4]). *Assume* $\phi(\alpha,\beta)$ *is* Σ^1_{2n} *(resp.* Π^1_{2n+1}*) then the predicate* "$r \leq \mu(\{\beta : \phi(\alpha,\beta)\})$" *is* Σ^1_{2n} *(resp.* Π^1_{2n+1}*) in* r *and* α. *(*r *is a rational number). The predicate* "$\{\beta : \phi(\alpha,\beta)\}$ *is not meager" is* Σ^1_{2n} *(resp.* Π^1_{2n+1}*) in* α.

The prewellordering technique will be essentially used here via the following well-known lemma.

LEMMA 1. *Let* $\leq$ *be a prewellordering on a set of reals* A; *if any initial segment of* $\leq$ *is of zero measure (resp. of the first category), then,— either* $\leq$ *is a non measurable subset of* $\mathcal{R}^2$ *(resp* $\leq$ *has not the Baire property)— or* A *is of zero measure (resp. of the first category).*

1. The structure of small sets

In this section, we state a theorem on small sets and discuss the situation for Π^1_1 and Σ^1_2 small sets.

THEOREM 2. *Assume projective determinacy, then there exist* Π^1_{2n+1} *and* Σ^1_{2n} *largest sets of zero measure (resp. of the first category).*

This is due independently to Kechris [5] and we give only a sketch of the proof for Σ^1_{2n} sets in the measure case.

Let $A(n,\alpha)$ be universal for Σ^1_{2n} sets, let ϕ a Σ^1_{2n} norm on A, $\leq_\varphi$ the corresponding prewellordering. We let M and M_n be defined by the following:

$$\alpha \in M \leftrightarrow \exists n \quad (A(n,\alpha) \,\&\, \mu(\{\beta : (n,\beta) \leq_\phi (n,\alpha)\}) = 0)$$

$$\alpha \in M_n \leftrightarrow \qquad A(n,\alpha) \,\&\, \mu(\{\beta \,; (n,\beta) \leq_\phi (n,\alpha)\} = 0.$$

M contains all Σ^1_{2n} zero measure sets and M is clearly Σ^1_{2n}; now, if $\mu(M) \neq 0$ then for some integer n, $\mu(M_n) \neq 0$; but M_n is a prewellordered set such that any initial segment is of zero measure; furthermore this prewellordering is Σ^1_{2n} as a subset of $\mathcal{R}^2$, therefore it is measurable; hence we get a contradiction by Lemma 1.

A. The Π_1^1 *case*

We will investigate the Π_1^1 case with a technique very close to forcing on admissible sets. To state our definitions we need the notion of code for a Borel set. Roughly speaking a code for a Borel set B is a real containing information on the way B is built from the open intervals with rational endpoints (for a precise definition see [11]).

DEFINITION 3. A real α is *weakly random* (resp. weakly Cohen generic) over L_δ if α does not belong to any zero measure (resp. first category) Borel set with a code in L_δ.

The reader should be aware of the fact that this is not the classical definition. Especially for $\delta = \omega_1$, our definition does not imply that $L\omega_1[\alpha]$ is admissible.

REMARK. From the existence of an "hyperarithmetic code" for Δ_1^1 sets it follows that the Δ_1^1 sets are the Borel sets with a code in $L\omega_1$.

THEOREM 3. *There is a largest* Π_1^1 *set of zero measure (resp. of the first category).*

The largest Π_1^1 *zero measure set is the set of* α's *such that either* $\omega_1^\alpha \neq \omega_1$ *or* α *is not weakly random over* $L\omega_1$.

The largest Π_1^1 *set of the first category is the set of* α's *such that either* $\omega_1^\alpha \neq \omega_1$ *or* α *is not weakly Cohen generic over* $L\omega_1$.

PROOF.[†] Sacks [10] has proved that $\{\alpha : \omega_1^\alpha \neq \omega_1\}$ is Π_1^1 of zero measure. The set of α's which are not weakly random over $L\omega_1$ is also Π_1^1 of zero measure.

To finish the proof, let us first recall the definition of a tree: a tree on $\omega \times \omega$ is a set of finite sequences of elements of $\omega \times \omega$ closed under subsequences. For any element s of T, say $s = \langle\langle m_0, n_0\rangle, \cdots, \langle m_k, n_k\rangle\rangle$ it is posible to define the sequence of "second coordinates" $\pi(s) = \langle n_0, \cdots, n_k\rangle$. If α is real, $T(\alpha)$ is the set of elements s of T such that $\pi(s)$ is an initial segment of α (i.e. $\pi(s) = \bar{\alpha}(j)$ for some j). $T(\alpha)$ is a tree on ω.

Now let A be a Π_1^1 set; it is well known that there is a recursive tree T on $\omega \times \omega$ such that $\alpha \in A \leftrightarrow T(\alpha)$ is well founded. Assume there is an element α in A such that $\omega_1^\alpha = \omega_1$; then $T(\alpha)$ has length $\xi < \omega_1$. But then, the set of β's such that $T(\beta)$ is a tree of length $\leqq \xi$ is a Δ_1^1 set which contains α as an element; therefore if A is of measure zero, α is not weakly L_{ω_1} random.

B. The Σ_2^1 *case*

It is not surprising that the problem of the existence of a largest Σ_2^1 zero

[†] The initial proof we gave in [13] used forcing.

measure set is connected with random reals (see [11] for a definition); the following theorem analyzes the notion of random real from the point of view of descriptive set theory.

THEOREM 4. 1) *The following two statements are equivalent:*
(i) *α is random over L.*
(ii) *$\delta_2^{1,\alpha} = \delta_2^1$ and α is weakly random over $L_{\delta_2^1}$.*
2) *Similarly, the following are equivalent:*
(i) *α is Cohen generic over L.*
(ii) *$\delta_2^{1,\alpha} = \delta_2^1$ and α is weakly Cohen generic over $L_{\delta_2^1}$.*

PROOF. We will restrict ourselves to the first case.

Assume first α is not random over L: then the following is true:

$\exists \gamma, \beta$ (γ is a well ordering & $\beta \in L_{|\gamma|}$ & β codes a Borel set of zero measure B & $\alpha \in B$).

It is not difficult to see that this statement is $\Sigma_2^1(\alpha)$; therefore, by the basis theorem γ can be chosen to be a $\Delta_2^1(\alpha)$ real; hence either $\delta_2^{1,\alpha} \neq \delta_2^1$ or the code β for B is in $L_{\delta_2^1}$ that is to say α is not weakly random over $L_{\delta_2^1}$.

Conversely, assume α is random over L. It is clear that α is weakly random over $L_{\delta_2^1}$. Suppose $\delta_2^{1,\alpha} > \delta_2^1$. Pick a real r in $L_{\delta_2^{1,\alpha}}$ such that r codes a well ordering of type δ_2^1. Assume r has the following definitions:

$$\langle m, n \rangle \in r \leftrightarrow \phi[\alpha, m, n]$$

$$\langle m, n \rangle \notin r \leftrightarrow \psi[\alpha, m, n]$$

where ψ and ϕ are Σ_2^1 relations. By Shoenfield absoluteness lemma, it is easy to see that ϕ and ψ also define r in $L[\alpha]$. Now we use forcing with Borel sets of positive measure: we let p be a forcing condition (i.e. a Borel set of positive measure with code in L) which forces

$$\forall m \forall n \, (m, n) \in \check{r} \leftrightarrow \phi(\boldsymbol{\alpha}, m, n) \, \&$$

$$\forall m \forall n \, (m, n) \notin \check{r} \leftrightarrow \psi(\boldsymbol{\alpha}, m, n),$$

where $\boldsymbol{\alpha}$ is a name for the generic real, and $\check{r}$ a name for r.
Let $v^{\mathscr{B}}$ be a Boolean extension of the universe with

$$[\![\aleph_1^L \text{ is countable}]\!] = 1.$$

For any random real β in p, that is for a set of positive measure [in $V^{\mathcal{B}}$] r is $\Delta^1_2(\beta)$. So in $V^{\mathcal{B}}$ we have

$$\mu(\{\beta : r \in \Delta^1_2(\beta)\}) > 0.$$

But this contradicts a result of Kechris ([5] theorem 3.1.1) which ensures that if $\mu(\{\beta : r \in \Delta^1_2(\beta)\}) > 0$ then r is Δ^1_2 (the reader will check that the proof of Kechris' theorem for the Δ^1_2 case can be carried through in $V^{\mathcal{B}}$ and uses no determinacy). We now turn to the largest Σ^1_2 zero measure set.

THEOREM 5.† *Assume one of the following equivalent properties:*

(i) *Almost all reals are random over L (resp. all reals except a set of the first category are Cohen generic over L).*

(ii) *Every Σ^1_2 set is Lebesgue measurable (resp. has the Baire property).*

(iii) *$\{\alpha : \delta^{1,\alpha}_2 \neq \delta^1_2\}$ is of zero measure (resp. of the first category);*

then,

(i) *There is a maximal zero measure Σ^1_2 set (resp. a maximal Σ^1_2 set of the first category).*

(ii) *This set is the set of non random reals over L (resp. non Cohen generic reals over L).*

PROOF (for the measure case). Let M be the set of reals which are not random over L; by the hypothesis $\mu(M) = 0$; furthermore M is Σ^1_2. Now if ϕ is a Σ^1_2 relation and contains a random real α then by Shoenfield absoluteness lemma $\phi(\alpha)$ holds in $L[\alpha]$; therefore there is a Borel set p with a code in L such that: $p \Vdash \phi(\boldsymbol{\alpha})$ where $\boldsymbol{\alpha}$ is a name for a random real). But then, for any random real in p that is for a set of β's of positive measure, $\phi(\beta)$ holds in $L[\beta]$, therefore in the universe, so $\{\beta : \phi(\beta)\}$ is not of zero measure.

We do not know if the converse of this theorem is true; nevertheless, we know the following result which is of some interest;

THEOREM 6. *Assume there is a maximal Π^1_3 set of zero measure (resp. of the first category), then almost all reals are random over L (resp. all reals except a set of the first category are Cohen generic over L).*

REMARK. The same result holds with Π^1_{2n+1} instead of Π^1_3.

PROOF. Let $\leq_L$ be the canonical well ordering of the reals in L and let $I(\alpha, \beta)$ be the statement $\{(\alpha)_n ; n \in \omega\}$ is the set of elements which are strictly smaller than β in the canonical well ordering. It is well known that $\leq_L$ and

† Parts of this theorem were certainly known to Solovay.

$I(\alpha,\beta)$ are uniformly Σ^1_2. Assume there is a maximal Π^1_3 set of zero measure A. We wish to prove that the set of reals which are random over L is included in A. If this is not true, one can define the first code β (in $\leq_L$) for a Borel set B of zero measure not included in A. We claim $A \cup B$ is Π^1_3 and derive a contradiction. Actually, we have:

$$\alpha \in A \cup B \leftrightarrow \alpha \in A \quad \text{or} \quad \forall\beta\forall\gamma\, [I(\gamma,\beta) \& \mu(\ulcorner\beta\urcorner)=0\ \&$$

$$\alpha \in \ulcorner\beta\urcorner \& \forall n\, (\alpha \notin \ulcorner(\gamma)_n\urcorner \text{ or } \mu(\ulcorner(\gamma)_n\urcorner) \neq 0) \Rightarrow \forall n (\mu(\ulcorner(\gamma)_n\urcorner) \neq 0 \text{ or } \ulcorner(\gamma)_n\urcorner \subseteq A)],$$

where $\ulcorner\beta\urcorner$ is the Borel set coded by β and $\alpha \in \ulcorner\beta\urcorner$ means: "β codes a Borel set B and $\alpha \in B$". So $A \cup B$ is Π^1_3.

Remark. By an argument of the same kind one can prove that if $V = L$ there is no maximal zero measure Σ^1_2 set.

Remark. Provided ZF is consistent, it is consistent to assume

- $V = L(r)$ where r is a real
- $\aleph_1^L = \aleph_1$
- There is a maximal Σ^1_2 set of zero measure.

To prove this result consider the generic extension of L obtained by forcing with closed sets of measure $\geq 1/2$. It is known that this set of conditions has the c.c.c. and that almost all reals in $L[G]$ are random.

2. The structure of large sets

In this section we refine some of the techniques used in [5] to analyze the structure of large sets. This enables us to provide a uniform treatment for the measure case and the category case, and to prove a conjecture of Kechris. Throughout this section, we assume projective determinacy.

A. *The kernel of a prewellordered large set*

Very often, we will restrict ourselves to the measure case; the reader will supply the analogous definitions and results in the category case.

Definition 4. Let A be a set of positive measure in a pointclass Γ, ϕ a Γ-norm on A, $\leq_\phi$ the corresponding prewellordering. The kernel of A with respect to ϕ, A^K_ϕ is defined by the following property:

$$\alpha \in A^K_\phi \leftrightarrow \mu(\{\beta : \beta \leq_\phi \alpha \& \alpha \leq_\phi \beta\}) > 0 \& \alpha \in A.$$

In other words the kernel is the set of elements which appear at some level in the prewellordering with "many" other elements.

$A - A^K_\phi$ is defined by:

$$\alpha \in A - A^K_\phi \leftrightarrow \mu(\{\beta : \beta \leqslant_\phi \alpha \,\&\, \alpha \leqslant_\phi \beta\}) = 0 \,\&\, \alpha \in A,$$

therefore it is not difficult to see for $\Gamma = \Sigma^1_{2n}$ or Π^1_{2n+1} both A^K_ϕ and $A - A^K_\phi$ are in Γ.

The following result shows that we don't lose very much by discarding the set $A - A^K_\phi$.

LEMMA 2. *Assume Γ is Σ^1_{2n} or Π^1_{2n+1}, then, $A - A^K_\phi$ is of zero measure.*

PROOF. The restriction of $\leqslant_\phi$ to $A - A^K_\phi$ is a prewellordering of $A - A^K_\phi$ and is a projective subset of $\mathcal{R} \times \mathcal{R}$ (therefore mesurable). So by Lemma 1, if $A - A^K_\phi$ is not of zero measure it has an initial segment of positive measure. Then, there exists an element of $A - A^k_\phi$, minimal with respect to $\leqslant_\phi$, such that if A^β is $\{\alpha : \alpha \leqslant_\phi \beta \,\&\, \alpha \in A - A^K_\phi\}$ we have $\mu(A^\beta) > 0$. Now if we let A_β be $\{\alpha : \alpha \leqslant_\phi \beta \,\&\, \beta \not\leqslant_\phi \alpha \,\&\, \alpha \in A - A^K_\phi\}$, A_β is a prewellordered set and—by the definition of β—any initial segment is of zero measure. So $\mu(A_\beta) = 0$ by Lemma 1. Finally, $\mu(A^\beta - A_\beta) > 0$ but $A^\beta - A_\beta$ is $\{\alpha : \alpha \leqslant_\phi \beta \,\&\, \beta \leqslant_\phi \alpha \,\&\, \alpha \in A\}$; thus we get $\beta \in A^K_\phi$; contradiction.

For the rest of the section we assume Γ is Σ^1_{2n} or Π^1_{2n+1}; we let A, ϕ be given; we write $\alpha \sim_\phi \beta$ for $(\alpha \leqslant_\phi \beta \,\&\, \beta \leqslant_\phi \alpha)$.

LEMMA 3. *Any real α in A^K_ϕ is equivalent* (mod $\sim_\phi$) *to a Δ real, (where Δ is Δ^1_{2n+1} if Γ is Π^1_{2n+1}, Δ^1_{2n} if Γ is Σ^1_{2n}).*

PROOF IN THE MEASURE CASE. Let $m = \mu(\{\beta : \beta \leqslant_\phi \alpha\})$

$$m' = \mu(\{\beta : \beta \leqslant_\phi \alpha \,\&\, \alpha \not\leqslant_\phi \beta\}).$$

Pick a rational number r such that $m < r < m'$ then

$$\{\gamma : \gamma \in A \,\&\, \mu(\{\beta : \beta \leqslant_\phi \gamma \,\&\, \gamma \not\leqslant_\phi \beta\}) < r \,\&\, \mu(\{\beta : \beta \leqslant_\phi \gamma\}) > r\}$$

is precisely $\{\gamma : \gamma \sim \alpha\}$, it is in Γ and of positive measure therefore it contains a Δ-real.

PROOF FOR THE CATEGORY CASE. Let s be a finite sequence of integers such that $\hat{s}-\{\beta:\beta\sim_{\phi}\alpha\}$ is meager. We can apply the same argument as before to the set $\{\gamma:\gamma\in A\ \&\ \hat{s}-\{\beta:\beta\leqslant_{\phi}\gamma\}$ is meager $\&\hat{s}-\{\beta:\beta\leqslant_{\phi}\gamma\&\gamma\not\leqslant_{\phi}\beta\}$ is not meager$\}$, which is $\{\gamma:\gamma\sim_{\phi}\alpha\}$.

COROLLARY. *Any equivalence class* (mod $\sim\phi$) *included in* A^{κ}_{ϕ} *is in* Δ.

We now make a systematic use of the device of labelling the equivalence classes of A^{κ}_{ϕ} introduced in the proof of the preceding lemma.

1. *The measure case*

We define $\lambda_0(r,\alpha)$ by:
r is a positive rational number $\&\alpha\in A\&\mu(\{\beta:\beta\leqslant_{\phi}\alpha\})>r\&\mu(\{\beta:\beta\leqslant_{\phi}\alpha\&\alpha\not\leqslant_{\phi}\beta\})\leqslant r$.
λ_0 is in Γ and the set of rationals such that there is some α with $\lambda_0(r,\alpha)$ is the set of r such that $r<\mu(A)$.
Now if $(r_n)_{n\geqslant 1}$ is a fixed recursive enumeration of the positive rational numbers we let

$$\lambda(n,\alpha)\leftrightarrow\lambda_0(r_n,\alpha).$$

2. *The category case*

In that case, we define $\lambda_0(s,\alpha)$ by:
s is a finite sequence of integers $\&\alpha\in A\&\hat{s}-\{\beta:\beta\leqslant_{\phi}\alpha\}$ is meager $\&\hat{s}-\{\beta:\beta\leqslant_{\phi}\alpha\&\alpha\not\leqslant_{\phi}\beta\}$ is not meager & $A-\hat{s}$ is not meager.
It is clear that λ_0 is in Γ and that the set of s such that $\exists\alpha\,\lambda_0(s,\alpha)$ is the set of s such that:

$A-\hat{s}$ is not meager
$\hat{s}-A$ is meager.

If $(s_n)_{n\geqslant 1}$ is a recursive enumeration of finite sequences of integers, we let $\lambda(n,\alpha)\leftrightarrow\lambda_0(s_n,\alpha)$.

DEFINITION 5. A set A has Δ measure if $M_A=\{r:r$ is a rational number & $r<\mu(A)\}$ is a Δ-real.

A set A has Δ-category if $C_A=\{s:s$ is a finite sequence of integers and $\hat{s}\cap A$ is not meager$\}$ is a Δ-real.

If A has Δ-category, then $\{s:\hat{s}-A$ is meager & $A-\hat{s}$ is not meager$\}$ is a Δ-real. Call this last set C'_A, we have:

$$s \in C'_A \leftrightarrow \forall s'(\hat{s}' \subseteq \hat{s} \rightarrow s' \in C_A) \& \exists s'(s' \in C_A \& \hat{s}' \cap \hat{s} = \emptyset).$$

THEOREM 7. *Assume A^K_ϕ is the kernel of A with respect to ϕ; let θ be the length of the restriction of ϕ to A^K_ϕ; the following are equivalent:*

(i) *A has Δ-measure.*

(ii) *A^K_ϕ is a Δ set.*

(iii) *$\theta < \delta$ (where δ is δ^1_{2n+1} in case $\Gamma = \Pi^1_{2n+1}$, δ^1_{2n} in case $\Gamma = \Sigma^1_{2n}$). Also, the following is true anyway: $\theta \leqslant \delta$.*

Similarly for category.

PROOF. (ii)$\rightarrow$(i) is an easy consequence of Theorem 1 (iv). We now prove (i)$\rightarrow$(ii); if A is of Δ-measure the set m_A of integers n such that $r_n < \mu(A)$ is a Δ-real; but A^K_ϕ has the following definitions:

$$\alpha \in A^K_\phi \leftrightarrow \exists n \in m_A \quad \exists \beta \in \Delta(\lambda(n,\beta) \& \alpha \sim_\phi \beta)$$

$$\alpha \notin A^K_\phi \leftrightarrow \forall n \in m_A \quad \exists \beta \in \Delta(\lambda(n,\beta) \& \neg(\alpha \sim_\phi \beta))$$

therefore, A^K_ϕ is a Δ-set.

In order to compute θ, we introduce the following prewellordering $\leqslant$ on m_A: $n \leqslant m \leftrightarrow \exists \alpha \in \Delta \;\; \exists \beta \in \Delta(\lambda(n,\alpha) \& \lambda(m,\beta) \& \alpha \leqslant_\phi \beta)$.
Actually $\leqslant$ is a Γ-prewellordering; to prove this, assume $m \in m_A$ then, for the measure case:

$$n \not\leqslant m \leftrightarrow \exists \alpha \in \Delta(\lambda(m,\alpha) \& \mu(\{\beta : \beta \leqslant_\phi \alpha\}) \geqslant r_n).$$

For the category case one should write instead:

$$n \not\leqslant m \leftrightarrow \exists \alpha \in \Delta(\lambda(m,\alpha) \& \hat{s}_n - \{\beta : \beta \leqslant_\phi \alpha\} \quad \text{is not meager}).$$

Now, if m_A is a Δ-real, $\leqslant$ is a Δ-prewellordering and therefore has length smaller than δ, which proves (i)$\rightarrow$(iii).
To draw the conclusion (iii)$\rightarrow$(i) as well as the extra assumption $\theta \leqslant \delta$, it remains to prove the following lemma:

LEMMA 4. *The length of a Γ-norm on a Γ subset of ω, x, which is not Δ is exactly δ.*

PROOF. Recall that Γ is Σ^1_{2n} or Π^1_{2n+1}. Let k a fixed element in x. If $\leqslant$ is the given Γ-norm on x and if $\theta < \delta$ is the length of $\leqslant$, let γ be a subset of $\omega \times \omega$

coding a Δ-well ordering of length θ. Define $I(k,\delta)$ to be: δ is an order preserving map from an initial segment of γ onto $(m : m \leqslant k)$. It is not difficult to see that $I(k,\delta)$ is in Γ as well as $\neg I(k,\delta)$. Now we let $\phi(n,k)$ to be $k \in x \& \forall\gamma [I(k,\delta) \Rightarrow \delta(n) \leqslant k \& k \leqslant \delta(n)]$ or equivalently $k \in x \& \exists\delta [I(k,\delta) \& \delta(n) \leqslant k \& k \leqslant \delta(n)]$ according as Γ is Π^1_{2n+1} or Σ^1_{2n}. For any n in the domain of γ, there is a k such that $k \in x \,\&\, \phi(n,k)$, so by the selection theorem there is an $f : \omega \to \omega$ in Δ such that $\forall n \in \mathrm{dom}(\gamma)(f(n) \in x \& \phi(n, f(n)))$. Then $k \in x \Leftrightarrow \exists n \in \mathrm{dom}(\gamma)(k \leqslant f(n))$, so x is in Δ.

The following result is an example of an application of the technique of kernels.

THEOREM 8. *Assume projective determinacy, then;*

1) *For any* Π^1_{2n+1} *(resp.* Σ^1_{2n}*) set* A*, there is a maximal* Π^1_{2n+1} *(resp.* Σ^1_{2n}*) set* A', $A \subseteq A'$, *with the same measure.*

2) *For any* Π^1_{2n+1} *(resp.* Σ^1_{2n}*) set* A*, there is a maximal* Π^1_{2n+1} *(resp.* Σ^1_{2n}*) set* A', $A' \subseteq A'$ *and* $A' - A$ *is meager.*

PROOF. Let M be the largest Π^1_{2n+1} set of zero measure. For a given Π^1_{2n+1} set A put $A' = A \cup M$. We claim any Π^1_{2n+1} set B with $\mu(B - A) = 0$ is in fact included in A'. Let ϕ be a norm on B; $B - B^K_\phi$ is Π^1_{2n+1} and of zero measure so it is included in M. Therefore if the claim is not true, there is an element β in B^K_ϕ not in A', consider $\{\alpha ; \alpha \sim_\phi \beta\} = B'$; B' is Δ^1_{2n+1} of positive measure; now $B' - A'$ is Σ^1_{2n+1} of zero measure so if C is the complement of $B' - A'$ and ψ is a norm on C we have $B' - A' \cap C^K_\phi = \varnothing$. But C^K_ϕ is a Δ^1_{2n+1} set of measure 1, so $B' - A'$ is included in a Δ^1_{2n+1} set of zero measure, therefore in M; contradiction.

The proof is the same for the Σ^1_{2n} Case.

B. The kernel with respect to a scale

Let $\psi = \{\psi_0, \cdots, \psi_1, \cdots.\}$ a Γ-scale on a fixed Γ set A with positive measure. We assume that ψ has the following property.

If $\psi_k(\alpha) = \psi_k(\alpha')$ then $\bar{\alpha}(k+1) = \bar{\alpha}'(k+1)$ and $\psi_i(\alpha) = \psi_i(\alpha')$ for any $i \leqslant k$. The standard procedure to go from a Γ-scale $(\phi_n)_{n \in \omega}$ to a scale with this property is to consider the lexicographic well ordering on $2(n+1)$-uples:

$$\langle \phi_0(\alpha), \alpha(0), \cdots, \phi_n(\alpha), \alpha(n) \rangle$$

and to define ψ accordingly. We leave the details to the reader.

What has been done in the preceding paragraph for A and ϕ, can be done (uniformly) for A and each ψ_i. Thus we can get a relation $\Lambda\ (i,n,\alpha)$ corresponding to $\lambda\ (n,\alpha)$ for each i;

From now on we assume A has Δ measure; then we notice that:

$$\neg\Lambda(i,n,\alpha)\leftrightarrow n\not\in m_A \quad\text{or}\quad \exists\beta\in\Delta(\neg(\beta\sim_{\psi_i}\alpha)\,\&\,\Lambda(i,n,\beta))$$

therefore $\Lambda(i,n,\alpha)$ is in fact in Δ. If we let $\Lambda(i,\alpha)$ to be the smallest integer n if it exists such that $\Lambda(i,n,\alpha)$ holds, 0 otherwise then Λ is a Δ-function. We now define a tree T on $\omega\times\omega$.

$$\langle n_0,n_1,\cdots,n_k:p_0,\cdots,p_k\rangle\in T\leftrightarrow$$

$$\exists\alpha\in\Delta(\bar\alpha(k+1)=\langle n_0,\cdots,n_k\rangle\,\&\,\forall j\ 0\leqslant j\leqslant k\rightarrow p_j=\Lambda(j,\alpha)\neq 0).$$

Another definition of T is

$$\exists\alpha\,(\bar\alpha(k+1)=\langle n_0,\cdots,n_k\rangle\,\&\,\forall j\ 0\leqslant j\leqslant k\rightarrow p_j=\Lambda(j,\alpha)\neq 0).$$

Now we claim if $p(T)=\{\alpha:\exists\beta\,(\beta,\alpha)$ is a branch through $T\}$ then

$$\bigcap_{i=1}^{\infty} A^K_{\psi_i}=A^K_{\psi}\subseteq p(T)\subseteq A.$$

Proof of Claim. Let α in A^K_{ψ}; for any n pick a Δ-real α_n such $\psi_n(\alpha_n)=\psi_n(\alpha)$ then

$$\bar\alpha(n+1)=\bar\alpha_n(n+1)$$

$$\psi_i(\alpha_n)=\psi_i(\alpha)\quad\text{for}\quad 0\leqslant i\leqslant n$$

therefore $\Lambda(i,\alpha_n)=\Lambda(i,\alpha)$ for $0\leqslant i\leqslant n$; finally if $\beta(i)=\Lambda(i,\alpha)$, (α,β) is a branch through T.

Now, let γ in $p[T]$, (γ,δ) be a branch through T; it follows from the definition of T that for any n there is a Δ-real α_n such that:

$$\begin{cases}\bar\gamma(n+1)=\bar\alpha_n(n+1)\\[2ex] \bar\delta(n+1)=\langle\Lambda(0,\alpha_n),\cdots,\Lambda(n,\alpha_n)\rangle.\end{cases}$$

Recall definition 2; $(\alpha_n)_{n\in\omega}$ is a sequence of reals such that $\lim_{n\to\infty}\alpha_n=\gamma$. For any $k\geqslant n$, $\phi_n(\alpha_k)=\phi_n(\alpha_n)$. Therefore, it follows that γ is in A.

T is a real (up to any coding device); from the above definitions it follows that if Γ is Π^1_{2n+1} T is a Δ^1_{2n+1}-real. Thus we have proved:

THEOREM 9. *Assume projective determinacy; let A be a Π^1_{2n+1} set with Δ^1_{2n+1} measure; then there is a Δ^1_{2n+1} tree T on $\omega \times \omega$ such that $p[T]$ is a subset of A with the same measure; similarly for category.*

This last result reduces the problem of approximating Π^1_{2n+1} sets to the easier one of approximating $\Sigma^1_1(\alpha)$ sets. As an example, we get a proof of the following theorem:

THEOREM 10† *Assume projective determinacy. Let A be a Π^1_{2n+1} set with positive Δ^1_{2n+1}-measure, then, there is a strongly Δ^1_{2n+1} $F\sigma$ subset of A with the same measure.*

Recall an $F\sigma$ set A is strongly Δ^1_n iff there is a Δ^1_n real T such that:

$$-\forall n\ (T)_n \quad \text{is a tree.}$$

$$-\alpha \in A \leftrightarrow \exists n\ (\forall m\ \bar{\alpha}(m) \in (T)_n).$$

PROOF. Apply the relativized version of the analogous theorem for Σ^1_2 sets ([5], th. 4.3.4.).

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